General second order conditions for extrema of functionals

Mahouton Norbert Hounkonnou[1,†] and Pascal Dkengne Sielenou[1,*]
[1] *University of Abomey-Calavi,*
International Chair in Mathematical Physics and Applications
(ICMPA–UNESCO Chair), 072 B.P. 50 Cotonou, Republic of Benin
E-mails: †norbert.hounkonnou@cimpa.uac.bj, *sielenou_alain@yahoo.fr

Abstract

We prove both necessary and sufficient second order conditions of extrema for variational problems involving any higher order continuously twice differentiable Lagrangians with multi-valued dependent functions of several variables. Our analysis is performed in the framework of the finite dimensional total jet space.

Key-words: Second order extrema conditions, total jet space, variational problem.

1 Introduction

The calculus of variations encompasses a very broad range of mathematical applications. The methods of variational analysis can be applied to an enormous variety of physical systems, whose equilibrium configurations minimize or maximize a suitable functional, which typically represents the potential energy of the system. The critical functions are characterized as solutions to a system of partial differential equations, known as the Euler-Lagrange equations derived from the variational principle. Each solution to the boundary value problem specified by the Euler-Lagrange equations subject to appropriate boundary conditions is thus a candidate extremizer for the variational problem. In many applications, the Euler-Lagrange boundary value problem suffices to single out the physically relevant solutions, and one needs not press onto the considerably more difficult second variation.

In general, the solutions to the Euler-Lagrange boundary value problem are critical functions for the variational problem, and hence include all (smooth) local and global extremizers. The determination of which solutions are genuine minima or maxima requires a further analysis of the positivity properties of the second variation. Indeed, as stated in [1], a complete analysis of the positive definiteness of the second variation of multi-dimensional variational problems is quite complicated, and still awaits a completely satisfactory resolution! This is thus a reason for which second order conditions of extrema is customary established only for at most two-dimensional variational problems involving first order Lagrangians [2, 3, 4]. The aim of this paper is to formulate the second order extrema conditions for multi-dimensional variational problems with any higher order Lagrangians.

2 Preliminaries: basic definitions, theorems and notations

This section, mainly based on [5, 6], addresses relevant definitions, theorems and notations playing a central role in the calculus of variations.

Let $x = (x^1, \dots, x^n)$ and $u = (u^1, \dots, u^m)$, $u = u(x)$, and $X \times U^{(s)}$, the space whose coordinates are denoted by $(x, u^{(s)})$, encompassing the independent variables x, the dependent variables u and their derivatives up to order s, $u^{(s)}$.

Definition 2.1 (Differential function)
A function f defined on $X \times U^{(s)}$ is called s-order differential function if it is locally analytic, i.e., locally expandable in a Taylor series with respect to all arguments.

Definition 2.2 (Total derivative operator)
Let f defined on $X \times U^{(s)}$ be an s-order differential function. The total derivative of f with respect to x^i is defined by:

$$D_{x^i} f = \frac{\partial f}{\partial x^i} + \sum_{j=1}^{m} \sum_{k_1+\cdots+k_n=0}^{s} u^j_{k_1 x^1 \cdots (k_i+1) x^i \cdots k_n x^n} \frac{\partial f}{\partial u^j_{k_1 x^1 \cdots k_n x^n}},$$

where

$$u^j_{k_1 x^1 \cdots k_n x^n} = \frac{\partial^{k_1+\cdots+k_n} u^j}{(\partial x^1)^{k_1} \cdots (\partial x^n)^{k_n}},$$

Definition 2.3 (Zeroth-Euler operator)
Let f defined on $X \times U^{(s)}$ be an s-order differential function. The zeroth-Euler operator (also called the variational derivative) of f is given by

$$\frac{\delta}{\delta u} f = \left(\frac{\delta}{\delta u^1} f, \cdots, \frac{\delta}{\delta u^m} f \right), \tag{2.1}$$

where for $j = 1, \cdots, m$

$$\frac{\delta}{\delta u^j} f = \sum_{k_1+\cdots+k_n=0}^{s} (-D_{x^1})^{k_1} \cdots (-D_{x^n})^{k_n} \frac{\partial f}{\partial u^j_{k_1 x^1 \cdots k_n x^n}}. \tag{2.2}$$

A variational problem consists in finding extrema of a functional J defined by

$$J[u] = \int_\Omega L\left(x, u^{(s)}(x)\right) dx, \tag{2.3}$$

where Ω is a connected open subset of X and L defined on $X \times U^{(s)}$ is an s-order differential function called the Lagrangian of the variational problem J. In general, a functional is a mapping that assigns to each element in some function space a real number, and a variational problem amounts to searching for functions which are an extremum (minimum, maximum) or saddle points of a given functional.

Theorem 2.1 *Let u be an extremal of J, then u satisfies the Euler-Lagrange equations*

$$\frac{\delta}{\delta u^j} L\left(x, u^{(s)}(x)\right) = 0, \quad j = 1, \cdots, m. \tag{2.4}$$

3 Necessary and sufficient conditions for extrema of functionals

In this section, we propose a definition of the total jet space and show that in this framework, the shape of second order conditions for extrema of functional whose Lagrangian includes multi-valued dependent functions of several variables remains the same as that of the second variation for a functional with first order one-dimensional scalar valued Lagrangian.

Consider X, an n-dimensional independent variables space, and $U = \bigotimes_{j=1}^{m} U^j$, an m-dimensional dependent variables space. Let $x = (x^1, \cdots, x^n) \in X$ and $u = (u^1, \cdots, u^m) \in U$ with $u^j \in U^j$. We define the jet-space $U^{(s)}$ as:

$$U^{(s)} := \bigotimes_{j=1}^{m} \left(\bigotimes_{l=0}^{s} U^j_{(l)} \right), \tag{3.1}$$

where $U^j_{(l)}$ is the set of all $p_l \equiv \begin{pmatrix} n+l-1 \\ l \end{pmatrix}$ distinct l-th order partial derivatives of u^j. We denote by $u^j_{(k)}$ the p_k-tuple of all k-order partial derivatives of u^j. The $u^j_{(k)}$ vector components are recursively obtained as follows:

i) $u^j_{(0)} = u^j$ and $u^j_{(1)} = (u^j_{x^1}, u^j_{x^2}, \cdots, u^j_{x^n})$.

ii) Assume that $u^j_{(k)}$ is known and form the tuples

$$\widetilde{u}^j_{(k+1)}(l) = \left(\frac{\partial}{\partial x^1} u^j_{(k)}[l], \frac{\partial}{\partial x^2} u^j_{(k)}[l], \cdots, \frac{\partial}{\partial x^n} u^j_{(k)}[l] \right), \quad l = 1, 2, \cdots, p_k;$$

$$\widetilde{u}^j_{(k+1)} = \left(\widetilde{u}^j_{(k+1)}(1), \widetilde{u}^j_{(k+1)}(2), \cdots, \widetilde{u}^j_{(k+1)}(p_k) \right),$$

where $u^j_{(k)}[l]$ is the l-th component of the vector $u^j_{(k)}$.

iii) The tuple $u^j_{(k+1)}$ is obtained from the tuple $\widetilde{u}^j_{(k+1)}$ in such a way to exclude all further components already written following the vector components order.

The simplest instances of the above situation are obtained when $n = 2$ and 3 as follows.

Example 3.1 *:*

- *For $n = 2$, $x = (x^1, x^2)$ and we have:*

$$u^j_{(1)} = \left(u^j_{x^1}, u^j_{x^2} \right),$$

and

$$\begin{aligned}
\widetilde{u}^j_{(2)}(1) &= \left(\frac{\partial}{\partial x^1} u^j_{(1)}[1], \frac{\partial}{\partial x^2} u^j_{(1)}[1] \right) = \left(u^j_{2x^1}, u^j_{x^1x^2} \right), \\
\widetilde{u}^j_{(2)}(2) &= \left(\frac{\partial}{\partial x^1} u^j_{(1)}[2], \frac{\partial}{\partial x^2} u^j_{(1)}[2] \right) = \left(u^j_{x^2x^1}, u^j_{2x^2} \right), \\
\widetilde{u}^j_{(2)} &= \left(\widetilde{u}^j_{(2)}(1), \widetilde{u}^j_{(2)}(2) \right) = \left(u^j_{2x^1}, u^j_{x^1x^2}, u^j_{x^2x^1}, u^j_{2x^2} \right), \\
u^j_{(2)} &= \left(u^j_{2x^1}, u^j_{x^1x^2}, \check{u}^j_{x^2x^1}, u^j_{2x^2} \right) = \left(u^j_{2x^1}, u^j_{x^1x^2}, u^j_{2x^2} \right).
\end{aligned}$$

- *For $n = 3$, $x = (x^1, x^2, x^3)$ and we get*

$$\begin{aligned}
u^j_{(2)} &= \left(u^j_{2x^1}, u^j_{x^1x^2}, u^j_{x^1x^3}, u^j_{2x^2}, u^j_{x^2x^3}, u^j_{2x^3} \right), \\
u^j_{(3)} &= \left(u^j_{3x^1}, u^j_{2x^1x^2}, u^j_{2x^1x^3}, u^j_{x^12x^2}, u^j_{x^1x^2x^3}, u^j_{x^12x^3}, u^j_{3x^2}, u^j_{2x^2x^3}, u^j_{x^22x^3}, u^j_{3x^3} \right),
\end{aligned}$$

for $k = 2$ and $k = 3$, respectively.

An element $u^{(s)}$ in the jet-space $U^{(s)}$ is the $m(1+p_1+p_2+\cdots+p_s) = m\begin{pmatrix} n+s \\ s \end{pmatrix}$-tuple defined by

$$u^{(s)} = \left(u^1_{(0)}, u^1_{(1)}, \cdots, u^1_{(s)}, u^2_{(0)}, u^2_{(1)}, \cdots, u^2_{(s)}, \cdots, u^m_{(0)}, u^m_{(1)}, \cdots, u^m_{(s)}\right).$$

Naturally, we have

$$u^{j(s)} = \left(u^j_{(0)}, u^j_{(1)}, u^j_{(2)}, \cdots, u^j_{(s)}\right), \quad j = 1, 2, \cdots, m.$$

Definition 3.1 *Consider an s-order variational problem defined by the functional*

$$J[u] = \int_\Omega L\left(x, u^{(s)}(x)\right) dx, \tag{3.2}$$

where Ω is an open subset of $\mathbb{R}^n$. Then, we define an $m \times m$-matrix A of second order partial derivatives of L by:

$$A = \left[A^{jj'}\right]_{1\leq j,\, j'\leq m} \quad \textit{with} \quad A^{jj'} = \left[A^{jj'}_{kk'}\right]_{0\leq k,\, k'\leq s}, \tag{3.3}$$

$$A^{jj'}_{kk'} = \left[\frac{\partial^2 L}{\partial u^j_{(k)}[h]\partial u^{j'}_{(k')}[h']}\right]_{\substack{1\leq h'\leq p_{k'} \\ 1\leq h\leq p_k,}}. \tag{3.4}$$

Example 3.2 *Let us construct the martrix A for particular values of the integers m, n, s. If $m = s = 2$, then*

$$A = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix} \quad \textit{with} \quad A^{jj'} = \begin{bmatrix} A^{jj'}_{00} & A^{jj'}_{01} & A^{jj'}_{02} \\ A^{jj'}_{10} & A^{jj'}_{11} & A^{jj'}_{12} \\ A^{jj'}_{20} & A^{jj'}_{21} & A^{jj'}_{22} \end{bmatrix}.$$

Explicitly, we obtain:

- *For $n = 1$, i.e. $x = x^1$:*

$$\begin{array}{lll} A^{jj'}_{00} = \frac{\partial^2 L}{\partial u^j \partial u^{j'}}, & A^{jj'}_{01} = \frac{\partial^2 L}{\partial u^j \partial u^{j'}_x}, & A^{jj'}_{02} = \frac{\partial^2 L}{\partial u^j \partial u^{j'}_{2x}}, \\ A^{jj'}_{10} = \frac{\partial^2 L}{\partial u^j_x \partial u^{j'}}, & A^{jj'}_{11} = \frac{\partial^2 L}{\partial u^j_x \partial u^{j'}_x}, & A^{jj'}_{12} = \frac{\partial^2 L}{\partial u^j_x \partial u^{j'}_{2x}}, \\ A^{jj'}_{20} = \frac{\partial^2 L}{\partial u^j_{2x} \partial u^{j'}}, & A^{jj'}_{21} = \frac{\partial^2 L}{\partial u^j_{2x} \partial u^{j'}_x}, & A^{jj'}_{22} = \frac{\partial^2 L}{\partial u^j_{2x} \partial u^{j'}_{2x}}, \end{array}$$

- *For $n = 2$, i.e. $x = (x^1, x^2)$:*

$$A^{jj'}_{00} = \frac{\partial^2 L}{\partial u^j \partial u^{j'}}, \quad A^{jj'}_{01} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j \partial u^{j'}_{x^1}} & \frac{\partial^2 L}{\partial u^j \partial u^{j'}_{x^2}} \end{pmatrix},$$

$$A^{jj'}_{02} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j \partial u^{j'}_{2x^1}} & \frac{\partial^2 L}{\partial u^j \partial u^{j'}_{x^1x^2}} & \frac{\partial^2 L}{\partial u^j \partial u^{j'}_{2x^2}} \end{pmatrix},$$

$$A_{10}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}} \\ \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}} \end{pmatrix}, \quad A_{11}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}_{x^1}} & \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}_{x^2}} \\ \frac{\partial^2 L}{\partial u^j_{x^2} \partial u^{j'}_{x^1}} & \frac{\partial^2 L}{\partial u^j_{x^2} \partial u^{j'}_{x^2}} \end{pmatrix},$$

$$A_{12}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}_{2x^1}} & \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}_{x^1 x^2}} & \frac{\partial^2 L}{\partial u^j_{x^1} \partial u^{j'}_{2x^2}} \\ \frac{\partial^2 L}{\partial u^j_{x^2} \partial u^{j'}_{2x^1}} & \frac{\partial^2 L}{\partial u^j_{x^2} \partial u^{j'}_{x^1 x^2}} & \frac{\partial^2 L}{\partial u^j_{x^2} \partial u^{j'}_{2x^2}} \end{pmatrix},$$

$$A_{20}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j_{2x^1} \partial u^{j'}} \\ \frac{\partial^2 L}{\partial u^j_{x^1 x^2} \partial u^{j'}} \\ \frac{\partial^2 L}{\partial u^j_{2x^2} \partial u^{j'}} \end{pmatrix}, \quad A_{21}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j_{2x^1} \partial u^{j'}_{x^1}} & \frac{\partial^2 L}{\partial u^j_{2x^1} \partial u^{j'}_{x^2}} \\ \frac{\partial^2 L}{\partial u^j_{x^1 x^2} \partial u^{j'}_{x^1}} & \frac{\partial^2 L}{\partial u^j_{x^1 x^2} \partial u^{j'}_{x^2}} \\ \frac{\partial^2 L}{\partial u^j_{2x^2} \partial u^{j'}_{x^1}} & \frac{\partial^2 L}{\partial u^j_{2x^2} \partial u^{j'}_{x^2}} \end{pmatrix},$$

$$A_{22}^{jj'} = \begin{pmatrix} \frac{\partial^2 L}{\partial u^j_{2x^1} \partial u^{j'}_{2x^1}} & \frac{\partial^2 L}{\partial u^j_{2x^1} \partial u^{j'}_{x^1 x^2}} & \frac{\partial^2 L}{\partial u^j_{2x^1} \partial u^{j'}_{2x^2}} \\ \frac{\partial^2 L}{\partial u^j_{x^1 x^2} \partial u^{j'}_{2x^1}} & \frac{\partial^2 L}{\partial u^j_{x^1 x^2} \partial u^{j'}_{x^1 x^2}} & \frac{\partial^2 L}{\partial u^j_{x^1 x^2} \partial u^{j'}_{2x^2}} \\ \frac{\partial^2 L}{\partial u^j_{2x^2} \partial u^{j'}_{2x^1}} & \frac{\partial^2 L}{\partial u^j_{2x^2} \partial u^{j'}_{x^1 x^2}} & \frac{\partial^2 L}{\partial u^j_{2x^2} \partial u^{j'}_{2x^2}} \end{pmatrix}.$$

Let $u = (u^1, \cdots, u^m) \in (\mathcal{C}^s(\Omega))^m$ and $\phi = (\phi^1, \cdots, \phi^m)$ with $\phi^j \in \mathcal{C}^\infty(\Omega)$, the set of continuously infinitely differentiable functions on Ω. Consider the function f defined by

$$f(\epsilon) = J[u + \epsilon\phi] = \int_\Omega L\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right) dx, \quad \forall\, \epsilon > 0.$$

Then, the first derivative of f yields

$$\begin{aligned} f'(\epsilon) &= \frac{d}{d\epsilon} J[u+\epsilon\phi] \\ &= \int_\Omega \sum_{j=1}^m \sum_{k=0}^s \sum_{h=1}^{p_k} \phi^j_{(k)}[h](x)\, \frac{\partial L\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right)}{\partial u^j_{(k)}[h]}\, dx^1 \cdots dx^n, \end{aligned}$$

or equivalently

$$f'(\epsilon) = \int_\Omega \sum_{j=1}^m \sum_{k_1+\cdots+k_n=0}^s \phi^j_{k_1 x^1 \cdots k_n x^n}(x)\, \frac{\partial L\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right)}{\partial u^j_{k_1 x^1 \cdots k_n x^n}}\, dx^1 \cdots dx^n.$$

The second derivative of f is given by

$$\begin{aligned} f''(\epsilon) &= \frac{d^2}{d\epsilon^2} J[u+\epsilon\phi] \\ &= \int_\Omega \sum_{j,j'=1}^m \sum_{k,k'=0}^s \sum_{h=1}^{p_k} \sum_{h'=1}^{p_{k'}} \phi^j_{(k)}[h](x) \phi^{j'}_{(k')}[h'](x) \frac{\partial^2 L\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right)}{\partial u^j_{(k)}[h]\, \partial u^{j'}_{(k')}[h']} dx \\ &= \int_\Omega \sum_{j,j'=1}^m \sum_{k,k'=0}^s \phi^j_{(k)}(x)\, A^{jj'}_{kk'}\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right)\, {}^t\phi^{j'}_{(k')}(x)\, dx \\ &= \int_\Omega \sum_{j,j'=1}^m \phi^{j(s)}(x)\, A^{jj'}\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right)\, {}^t\phi^{j'(s)}\, dx \\ &= \int_\Omega \phi^{(s)}(x)\, A\left(x, u^{(s)}(x) + \epsilon\phi^{(s)}(x)\right)\, {}^t\phi^{(s)}(x)\, dx \end{aligned}$$

leading to the simplified form

$$
\begin{aligned}
f''(\epsilon) &= \int_\Omega \sum_{j,j'=1}^{m} \sum_{k_1+\cdots+k_n=0}^{s} \sum_{k'_1+\cdots+k'_n=0}^{s} \frac{\partial^2 L\left(x, u^{(s)}(x)+\epsilon\phi^{(s)}(x)\right)}{\partial u^j_{k_1x^1\cdots k_nx^n}\partial u^{j'}_{k'_1x^1\cdots k'_nx^n}} \\
&\times \ \phi^j_{k_1x^1\cdots k_nx^n}(x)\,\phi^{j'}_{k'_1x^1\cdots k'_nx^n}(x)\,dx.
\end{aligned}
$$

Provided all these definitions, we may now state the main results of this work.

Theorem 3.1 (Sufficient condition of extrema)
Let W be an open subspace of the Fréchet space $(\mathcal{C}^s(\Omega))^m$ and $J : W \to \mathbb{R}$ a functional defined by (3.2), continuously twice differentiable at $\overline{u} \in W$ such that

$$
\frac{\delta}{\delta u^j} L\left(x, \overline{u}^{(s)}(x)\right) = 0, \quad j = 1, \cdots, m, \tag{3.5}
$$

i.e. the function $\overline{u}$ is a critical point of the functional J. Then, J admits a local minimum (resp. maximum) at the critical point $\overline{u}$ if the corresponding matrix $A\left(x, \overline{u}^{(s)}(x)\right)$ defined by (3.3) is positive (resp. negative) definite for all $x \in \Omega$.

Proof. For all $\phi = (\phi^1, \cdots, \phi^m)$ with $\phi^j \in \mathcal{C}^\infty(\Omega)$, the second derivative of the real valued function f, defined by $f(\epsilon) = J[\overline{u} + \epsilon\phi]$, at $\epsilon = 0$ is

$$
\begin{aligned}
f''(0) &= \frac{d^2}{d\epsilon^2} J[\overline{u} + \epsilon\phi]|_{\epsilon=0} \\
&= \int_\Omega \phi^{(s)}(x)\, A\left(x, \overline{u}^{(s)}(x)\right)\, {}^t\phi^{(s)}(x)\,dx.
\end{aligned}
$$

Thus, if the matrix $A\left(x, \overline{u}^{(s)}(x)\right)$ defined by (3.3) is positive (resp. negative) definite for all $x \in \Omega$, then $J''(\overline{u}) = f''(0) > 0$ (resp. < 0), i.e. the function $\overline{u}$ is a local minimum (resp. maximum) point for the functional J. ■

Assume now that $\Omega = \prod_{i=1}^n]a_i,\, b_i[$ with $a_i,\, b_i \in \mathbb{R}$.

Theorem 3.2 (Necessary condition of extrema)
Let W be an open subspace of the Fréchet space $(\mathcal{C}^s(\Omega))^m$ and $J : W \to \mathbb{R}$ a functional defined by (3.2), continuously twice differentiable on W. Let $\overline{u} \in W$ be a local minimum (resp. maximum) point of J. Then for all $x \in \Omega$,

$$
B_l(x) \equiv \sum_{j,j'=1}^{m} \sum_{h,h'=1}^{p_l} \frac{\partial^2 L\left(x, \overline{u}^{(s)}(x)\right)}{\partial u^j_{(l)}[h]\partial u^{j'}_{(l)}[h']} \geq 0 \ (\textit{resp.} \leq 0), \quad l = 0, 1, 2, \cdots, s. \tag{3.6}
$$

Proof. Let $\overline{x} = (\overline{x}^1, \cdots, \overline{x}^n) \in \Omega$. Then there exists $r_0 > 0$ such that the ball of radius r_0 centered at $\overline{x}$, $B(\overline{x}; r_0) \subset \Omega$. Choosing $0 < \epsilon < \frac{r_0}{2n}$ implies $\prod_{i=1}^n \left[\overline{x}^i - \epsilon,\, \overline{x}^i + \epsilon\right] \subset B(\overline{x}; r_0)$. For $l = 0, 1, 2, \cdots, s$ we define some particular functions ϕ_l by

$$
\phi_l(x) = \sum_{i=1}^{n} \phi_{l,i}(x^i), \quad \forall\, x \in \prod_{i=1}^{n} \left[\overline{x}^i - \epsilon,\, \overline{x}^i + \epsilon\right], \tag{3.7}
$$

where for $i = 1, 2, \cdots, n$

$$
\phi_{l,i}(x^i) = \begin{cases} 0, & a_i \leq x^i \leq \overline{x}^i - \epsilon \\ 1 - \frac{1}{\epsilon}(x^i - \overline{x}^i)^l \operatorname{sign}(x^i - \overline{x}^i), & \overline{x}^i - \epsilon \leq x^i \leq \overline{x}^i + \epsilon \\ 0, & \overline{x}^i + \epsilon \leq x^i \leq b_i. \end{cases} \tag{3.8}
$$

Clearly, the functions ϕ_l are continuously infinitely differentiable everywhere except at the end points of its domain. Consider the real valued function f defined by

$$\begin{aligned} f(\widetilde{\epsilon}) &= \widetilde{J}[\overline{u}+\widetilde{\epsilon}\phi] \\ &= \int_{\prod_{i=1}^{n}\left[\overline{x}^i-\epsilon,\,\overline{x}^i+\epsilon\right]} L\left(x, \overline{u}^{(s)}(x)+\widetilde{\epsilon}\phi^{(s)}(x)\right) dx, \quad \forall \widetilde{\epsilon}>0, \end{aligned} \tag{3.9}$$

with $\phi = (\phi^1, \cdots, \phi^m)$, where the real valued functions ϕ^j are continuously infinitely differentiable everywhere except at the end points of the domain $\prod_{i=1}^{n}\left[\overline{x}^i-\epsilon,\,\overline{x}^i+\epsilon\right]$. Then,

$$\begin{aligned} f''(0) &= \frac{d^2}{d\widetilde{\epsilon}^2} J[\overline{u}+\widetilde{\epsilon}\phi]|_{\widetilde{\epsilon}=0} \\ &= \int_{\prod_{i=1}^{n}\left[\overline{x}^i-\epsilon,\,\overline{x}^i+\epsilon\right]} \sum_{j,j'=1}^{m}\sum_{k,k'=0}^{s}\sum_{h=1}^{p_k}\sum_{h'=1}^{p_{k'}} \frac{\partial^2 L\left(x, \overline{u}^{(s)}(x)\right)}{\partial u^j_{(k)}[h]\,\partial u^{j'}_{(k')}[h']} \\ &\times\ \phi^j_{(k)}[h](x)\phi^{j'}_{(k')}[h'](x)dx. \end{aligned} \tag{3.10}$$

Substitute in (3.10) the functions ϕ^j and $\phi^{j'}$, $j, j' = 1, 2, \cdots, m$ by the same function ϕ_l defined by (3.7) and let $\epsilon \to 0$. We obtain, after computation and simplification,

$$\begin{aligned} f''(0) &= \lim_{\epsilon\to 0}\left(\int_{\prod_{i=1}^{n}\left[\overline{x}^i-\epsilon,\,\overline{x}^i+\epsilon\right]} \sum_{j,j'=1}^{m}\sum_{h=1}^{p_l}\sum_{h'=1}^{p_l} \phi^j_{(l)}[h](x)\phi^{j'}_{(l)}[h'](x)\frac{\partial^2 L\left(x, \overline{u}^{(s)}(x)\right)}{\partial u^j_{(k)}[h]\,\partial u^{j'}_{(k')}[h']}dx\right) \\ &= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^2}\int_{\prod_{i=1}^{n}\left[\overline{x}^i-\epsilon,\,\overline{x}^i+\epsilon\right]} B_l(x)\ dx^1\cdots dx^n\right). \end{aligned} \tag{3.11}$$

Since the function $\overline{u}$ is a local minimum (resp. maximum) for the functional J, then we must have

$$f''(0) = J''(\overline{u}) \geq 0 \quad \text{(resp. } \leq 0). \tag{3.12}$$

Taking into account the equality (3.11), the conditions (3.12) lead to the inequalities

$$B_l(\overline{x}) \geq 0 \quad \text{(resp. } \leq 0). \tag{3.13}$$

This end the proof since $\overline{x}$ is an arbitrary point in the open subset Ω. ■

Acknowledgments

This work is partially supported by the ICTP through the OEA-ICMPA-Prj-15. The ICMPA is in partnership with the Daniel Iagolnitzer Foundation (DIF), France.

References

[1] P. J. Olver, The calculus of variations. *www.math.umn.edu/ olver/am_/cvz.pdf*.

[2] I. Gelfand and S. Fomin, Calculus of Variations. *Prentice-Hall* (1963).

[3] C. Fox, An introduction to the calculus of variations. *Oxford university press* (1950).

[4] L. Solanilla, A. Baquero and W. Naranjo, Second order conditions for extrema of functionals defined on regular surfaces. *Balkan Journal of Geometry and its Applications* **8** (2003), pp. 97-104.

[5] P. J. Olver, Applications of Lie groups to differential equations. *Springer, New york* (1993).

[6] L. D. Poole, Symbolic computation of conservation laws of nonlinear partial differential equations using homotopy operators. *PhD Thesis, Colodaro School of Mines* (2007).